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COMMENT

On AB percolation on bipartite graphs

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Abstract. A method is presented for determining lower bounds for the AB percolation critical probability of bipartite graphs. For some graphs, the method shows that AB percolation is impossible in an interval containing $p=\frac{1}{2}$. The results improve those of Appel and Wierman by removing symmetry and periodicity restrictions, and lend support to a conjecture of Halley.

1. Introduction

Let the vertices of an infinite graph G be independently labelled A with probability p and B with probability $1-p$. Connect adjacent vertices of G which have opposite labels with a bond, while adjacent vertices with the same label are not bonded together. This variant of the classical site percolation model was introduced by Mai and Halley (1980), as 'AB percolation', for the study of gelation processes, and independently by Turban (1983) and Sevšek *et al* (1983), as 'antipercolation', for the study of antiferromagnetism. The object of study is the probability distribution of the size of clusters of vertices which are connected by AB bonds.

The first step in the study of AB percolation is to determine if it is possible to have infinite AB clusters on a specified graph G and, if so, to determine the set of values of p for which infinite AB clusters exist. Appel and Wierman (1987) proved that AB percolation is impossible on a class of bipartite graphs (including the square and hexagonal lattices), partially verifying a conjecture of Halley (1983). Wierman and Appel (1987) proved that infinite AB percolation exists on the triangular lattice when $p\in[0.497, 0.503]$, which should be compared with the interval $[0.2145, 0.7855]$ obtained by Monte Carlo simulation by Mai and Halley (1980). Wierman (1988a) proved that the critical probability of AB percolation on the triangular lattice is equal to the site percolation critical probability of the triangular lattice with nearest- and next-nearest-neighbour bonds. The paper also shows that AB percolation occurs on any graph with site percolation critical probability strictly less than $\frac{1}{2}$. Wierman (1988b) proved that if G is in a class of close-packed graphs, then the AB percolation critical probability of G is equal to the site percolation critical probability of a related graph G_2. G_2 has the same vertex set as G, and a pair of vertices are adjacent in G_2 if they are connected by a path of length two in G. For the triangular lattice T, the graph T_2 is the same as T with both nearest- and next-nearest-neighbour bonds. Wierman (1988b) also observes that AB percolation occurs on the triangular lattice when $p\in[0.3473, 0.6527]$.

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This comment focuses on the study of AB percolation on bipartite graphs. It provides an alternative to the proof of Appel and Wierman (1987) that AB percolation does not exist on certain bipartite graphs. It shows that the symmetry and periodicity conditions in the result of Appel and Wierman (1987) are not needed. The method here proves that AB percolation cannot occur for an interval of values containing $\frac{1}{2}$ on graphs where the site percolation critical probability corresponding to one bipartition set is at least $\frac{1}{2}$. In principle, the method provides lower bounds for the AB percolation critical probability of bipartite graphs, although some knowledge of classical site percolation critical probabilities is necessary for calculation of the bounds. One consequence is that the AB percolation critical probability of a bipartite graph G is strictly greater than the classical site percolation critical probability of G_2, in contrast to the equality obtained by Wierman (1988b) for close-packed graphs.

2. Definitions

A graph G is bipartite if there exists a partition of its vertex set into two sets V_1 and V_2 such that every edge of G has one endpoint in V_1 and one endpoint in V_2. Note that any path on a bipartite graph passes alternately through vertices of V_1 and V_2.

An edge of G is an AB *bond* if the endpoints of the edge have different labels. A path is an AB *path* if all its edges are AB bonds. The AB *cluster* containing a vertex v, denoted W_v^{AB}, is the set of all vertices which may be reached from v through an AB path. The number of vertices in W_v^{AB} is denoted by $|W_v^{\mathrm{AB}}|$.

Define the AB percolation probability by

$$\theta_v^{\mathrm{AB}}(p, G) = P_p(|W_v^{\mathrm{AB}}| = +\infty).$$

Note that AB clusters are unchanged if the label of every vertex is changed, while the parameter of the model is changed from p to $1-p$. Thus

$$\theta_v^{\mathrm{AB}}(p, G) = \theta_v^{\mathrm{AB}}(1-p, G)$$

for all $p \in [0, 1]$, so the AB percolation probability function is symmetric about $\frac{1}{2}$.

While the value of $\theta_v^{\mathrm{AB}}(p, G)$ may depend on the vertex v, the set of values p for which $\theta_v^{\mathrm{AB}}(p, G) > 0$ is independent of the choice of v if G is a connected graph. Thus, for a connected graph G and an arbitrary vertex v, we define the AB percolation critical probability by

$$p_H^{\mathrm{AB}}(G) = \inf\{p : \theta_v^{\mathrm{AB}}(p, G) > 0\}.$$

Note that, by symmetry, if AB percolation occurs, $p_H^{\mathrm{AB}}(G) \leqslant \frac{1}{2}$.

If G is bipartite and the labels are reversed on either bipartition set, an AB cluster becomes a monochromatic cluster in a multiparameter site percolation model with parameters p and $1-p$ corresponding to probabilities that sites are open in the two bipartition sets. Thus, AB percolation occurs on G if and only if there exists $p \in [0, 1]$ so that both $(p, 1-p)$ and $(1-p, p)$ are in the percolative region of the parameter space for the multiparameter model. Although this gives a possible method for determining the existence of AB percolation on bipartite graphs, few multiparameter models are solved, so little information is gained from this fact.

3. Results

Let G be a bipartite graph with bipartition sets V_1 and V_2. For each V_i, construct a graph H_i with vertex set V_i, such that vertices u and v are adjacent in H_i if and only if u and v are adjacent to a common vertex in G. We will denote the classical site percolation critical probability of H_i by $p_H(H_i)$. If there is an infinite AB path in G, then there must be an infinite path of vertices labelled A in one H_i and an infinite path of vertices labelled B in the other. Then, either $p \geq p_H(H_1)$ and $1-p \geq p_H(H_2)$, or $p \geq p_H(H_2)$ and $1-p \geq p_H(H_1)$, so we must have $p_H(H_1)+p_H(H_2) \leq 1$. Thus, AB percolation is impossible if the sum of these critical probabilities is greater than one. In the case when the sum is equal to one, the impossibility of AB percolation was proved when certain symmetry and periodicity conditions hold (see Appel and Wierman 1987). In the following, we show that no symmetry or periodicity conditions are needed when the sum is equal to one.

For simplicity, we describe the method for a bipartite graph in which all vertices of V_1 have degree a in G and all vertices of V_2 have degree b in G. The results described below are all valid whenever the maximum vertex degree of G is finite, but the expressions become more complicated.

Let G^+ denote the graph obtained by inserting a vertex on each edge of G. We will essentially replace each vertex $v \in V_2$ by the set of vertices inserted on edges incident to v, so we refer to the inserted vertices as *parts* of v.

We construct coupled AB percolation configurations on G and G^+ and a site percolation configuration on H_1 as follows.

(i) Generate an AB configuration on G^+. Give each vertex of H_1 the label A with probability p, and each part of each vertex of H_2 the label B with probability $(1-p)^{1/b}$, independently.

(ii) Construct an AB configuration on G. Give each vertex of H_1 the label A if it is labelled A in G^+, and B otherwise. Give each vertex of H_2 the label B if all its parts are labelled B in G^+, and A otherwise. Each vertex of G has probability p of being labelled A, so this is a standard AB percolation model on G.

(iii) Construct a site percolation model on H_1. Let each vertex of H_1 be open if it is labelled A in G^+ and at least two of the parts of vertices on incident edges are labelled B.

By construction, each vertex of H_1 is open with probability

$$p\left(\sum_{i=2}^{a}\binom{a}{i}(1-p)^{i/b}[1-(1-p)^{1/b}]^{a-i}\right).$$

We can also reverse the roles of the bipartition sets, viewing the inserted vertices as parts of vertices of V_1. In this case, we obtain a site percolation model in which each vertex is open with probability

$$p\left(\sum_{i=2}^{b}\binom{b}{i}(1-p)^{i/a}[1-(1-p)^{1/a}]^{b-i}\right).$$

Suppose $p > p_H^{\mathrm{AB}}(G)$. Then, with positive probability, there is an infinite AB path $v_0, v_1, v_2, \ldots,$ in G. In this case, either $v_1, v_3, v_5, \ldots,$ is an infinite A path in H_1 and $v_2, v_4, v_6, \ldots,$ is an infinite B path in H_2, or vice versa. These give rise to open paths in the site percolation models on H_1 and H_2. Thus, either

$$p\left(\sum_{i=2}^{a}\binom{a}{i}(1-p)^{i/b}[1-(1-p)^{1/b}]^{a-i}\right) \geq p_H(H_1)$$

and

$$(1-p)\left(\sum_{i=2}^{b}\binom{b}{i}p^{i/a}[1-p^{1/a}]^{b-i}\right)\geqslant p_H(H_2)$$

or

$$p\left(\sum_{i=2}^{b}\binom{b}{i}(1-p)^{i/a}[1-(1-p)^{1/a}]^{b-i}\right)\geqslant p_H(H_2)$$

and

$$(1-p)\left(\sum_{i=2}^{a}\binom{a}{i}p^{i/b}[1-p^{1/b}]^{a-i}\right)\geqslant p_H(H_1).$$

Note that the second pair of inequalities may be obtained from the first by replacing p by $1-p$, so we need only consider the first pair and use symmetry of the AB percolation probability about $\frac{1}{2}$.

Let p_1 denote the minimum solution to the first inequality, and p_2 denote the maximum solution to the second inequality. Note that, in each inquality, p or $1-p$ is multipled by a factor which is strictly smaller than one (when $0<p<1$). Thus, since the left sides are continuous functions of p, strict inequalities hold for the solutions:

$$p_1>p_H(H_1)$$

and

$$1-p_2>p_H(H_2).$$

From this observation, we obtain theorem 1.

Theorem 1. If G is bipartite, with bipartition set graphs H_1 and H_2 such that

$$p_H(H_1)+p_H(H_2)\geqslant 1$$

and the maximum vertex degree of G is finite, then

$$\theta_v^{AB}(p,G)=0$$

for all $p\in[0,1]$.

Note that the result requires no symmetry or periodicity conditions (as in Appel and Wierman (1987)) and the result is valid in any dimension.

The first pair of inequalities requires that $p>p_1>p_H(H_1)$, while the second set requires that $p>p_2>p_H(H_2)$. Since at least one pair of inequalities must be satisfied, AB percolation cannot occur unless $p\geqslant\min\{p_1,p_2\}>\min\{p_H(H_1),p_H(H_2)\}$. We may use this fact to compute lower bounds for the AB percolation critical probability, as illustrated in the example below. Since, for a bipartite graph, the site percolation critical probability of the graph H_2 is the minimum appearing on the right of this inequality, we have theorem 2.

Theorem 2. If G is bipartite and has finite maximum vertex degree, then

$$p_H^{AB}(G)>p_H(G_2).$$

Note further that the first pair of inequalities can be satisfied only if $p \geq p_1 > p_H(H_1)$ and the second pair can be satisfied only if $1-p \geq 1-p_1 > p_H(H_1)$. Since at least one pair must be satisfied to have AB percolation, when $p_H(H_1) > \frac{1}{2}$ infinite AB percolation clusters cannot occur in the interval $(1-p_1, p_1)$, which contains $\frac{1}{2}$ (and similarly, if $p_H(H_2) > \frac{1}{2}$). This behaviour holds for the following example and the dice lattice, which both have the triangular lattice as one of the corresponding graphs H_i.

To illustrate the computations and phenomena discussed above, we present one example. Consider the triangular lattice with an additional vertex inserted at the midpoint of each edge. This graph is bipartite, with the bipartition sets being the vertices of the triangular lattice and the set of inserted vertices. Furthermore, the resulting graphs H_1 and H_2 are the triangular lattice and the matching lattice of the Kagomé lattice, for which the site percolation critical probabilities are both known exactly. In this case, one set of inequalities becomes

$$p(1-p)^{1/3} \geq 0.3473$$

and

$$(1-p)\left(\sum_{i=2}^{6}\binom{6}{i}p^{i/2}[1-p^{1/2}]^{6-i}\right) \geq \tfrac{1}{2}$$

which are both satisfied when $p \in (0.4153, 0.4948)$. Thus, the AB percolation critical probability is at least 0.4153 and AB percolation is impossible in the interval (0.4948, 0.5052). However, we cannot show that AB percolation is impossible on this graph, which would be implied by Halley's (1983) conjecture.

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